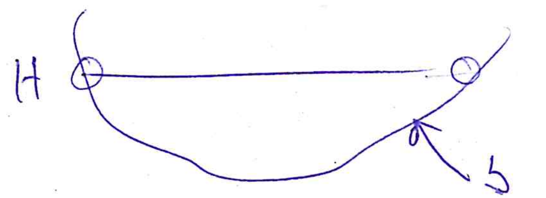
We discussed the conservative system

$$\frac{\partial Q}{\partial t} + \partial_x \underline{f}(Q) = \underline{S} \quad \text{with} \quad \underline{S} = \underline{0}$$

Then we studied the SWE at lake at rest



$u = 0$

$$Q = \begin{pmatrix} h \\ hu \end{pmatrix} \quad f = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 \\ -gh\frac{\partial b}{\partial x} \end{pmatrix}$$

By changing

$$\underline{Q} = \begin{pmatrix} h \\ hu \\ b \end{pmatrix} \quad \underline{f} = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \\ 0 \end{pmatrix}$$

we wrote

$$\underline{S} = \begin{pmatrix} 0 \\ -gh\frac{\partial b}{\partial x} \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -gh \\ 0 & 0 & 0 \end{pmatrix}}_{\underline{\underline{-NC}}} \partial_x (Q)$$

we introduced the non-conservative PDE system

$$\frac{\partial Q}{\partial t} + \partial_x (f(Q)) + \underline{\underline{NC}} \cdot \partial_x Q = \underline{S} = \underline{0}$$

$$\Leftrightarrow \frac{\partial}{\partial t} Q + \underbrace{\left(\underline{\underline{\frac{\partial f}{\partial Q}}} + \underline{\underline{NC}} \right)}_{\equiv \underline{\underline{A}}} \cdot \partial_x Q = \underline{0} \quad (I)$$

$\uparrow$

$$\partial_x f(Q) = \frac{\partial f}{\partial Q} \cdot \frac{\partial Q}{\partial x}$$

~~Conside a~~ The FVM:

Consider a space-time control volume (CV)

$[x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$ and integrate (I)

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^n}^{t^{n+1}} \partial_t \underline{Q} + \underline{\underline{A(Q)}} \cdot \frac{\partial Q}{\partial x} \, dt\,dx = 0$$

$$\Leftrightarrow \int_{x_{i-1/2}}^{x_{i+1/2}} Q(x, t^{n+1}) - Q(x, t^n) \, dx$$

$$+ \underbrace{\int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \underline{\underline{A(Q)}} \cdot \frac{\partial Q}{\partial x} \, dx\,dt}_{\equiv ?}$$

Recall that in case $\underline{\underline{NC}} = 0$

$$\Rightarrow A = \frac{\partial f}{\partial Q}$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \underbrace{\frac{\partial f}{\partial Q} \frac{\partial Q}{\partial x}}_{= \frac{\partial f}{\partial x}} \, dx \, ,$$

$$= f(x_{i+1/2}, t) - f(x_{i-1/2}, t)$$

We now try to make sense out of the product

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \underline{\underline{A(Q)}} \cdot \frac{\partial Q}{\partial x}\, dx .$$

Roe's method :

We assume $\underline{\underline{NC}} = \underline{\underline{0}}$.

Now we have 2 version of our PDE:

$$(II) \quad \partial_t Q + A(Q) \frac{\partial Q}{\partial x} = 0$$

$$(III) \quad \Leftrightarrow \partial_t Q + \frac{\partial f(Q)}{\partial x} = 0$$

We introduce the linearized ~~syst~~ RP.

$\tilde{Q}$ is our lin. ~~po~~ state

$$(\text{lin. R.P}) \begin{cases} \partial_t \overline{Q} + \overline{A}(\tilde{Q}) \dfrac{\partial \overline{Q}}{\partial x} = 0 \\ \overline{Q}(x, t=0) = \begin{cases} Q_L & x < 0 \\ Q_R & x > 0 \end{cases} \end{cases}$$

same data as in (orig. RP)

For (III), we had the RP.

(orig. R.P) $\begin{cases} \partial_t Q + \cancel{A(Q)} \dfrac{\partial f(Q)}{\partial x} = 0 \\ Q(t=0,x) = \begin{cases} Q_L & x<0 \\ Q_R & x>0 \end{cases} \end{cases}$

The idea of Roe's method is to use a <u>proper linearization</u> of A.

The problem will be to define $\tilde{Q} = Q^{Roe}$.

We want to have the following properties:

- Hyperbolicity: $\bar{A} \in \mathbb{R}^{m \times m}$ for $\tilde{Q} \in \mathbb{R}^m$ has m real ~~eigenvectors~~ eigenvalues and m linearly independent eigenvectors.

- Consistency: $\bar{A}(Q_L, Q_R) \underset{\underset{Q_L = Q_R = Q}{\uparrow}}{=} \bar{A}(Q,Q) = \dfrac{\partial f}{\partial Q}(Q)$

- Conservation:

$$\bar{A}(Q_L, Q_R) \cdot (Q_R - Q_L)$$
$$= f(Q_R) - f(Q_L).$$

~~Goal~~ Goal: given (lin. RP) and (orig. RP),

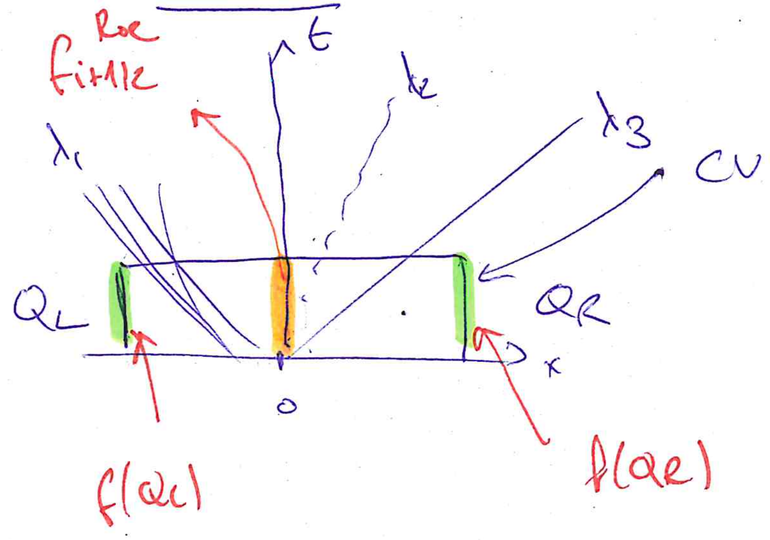
construct $f^{Roe}_{i+1/2}$ in order to use our FVM:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(f^{Roe}_{i+1/2} - f^{Roe}_{i-1/2}\right)$$

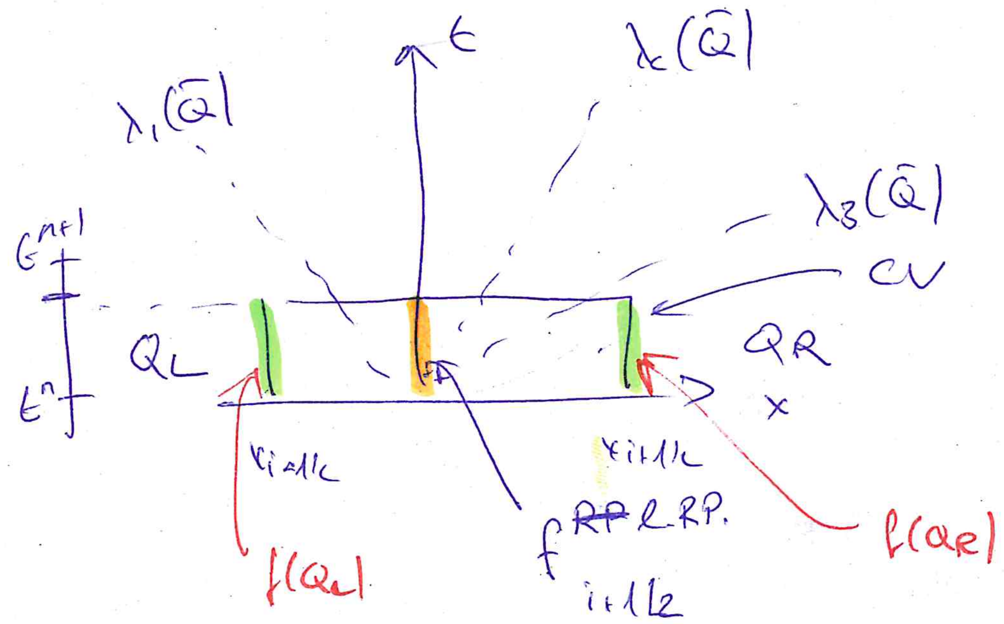
Remark:

$f^{Roe}_{i+1/2}$ should be defined in terms of our lin. matrix $\bar{A}(\tilde{Q})$.

orig. RP.



lin. RP :



Assumption (of Roe):

The integrals of (orig. RP) and the (lin. RP) over the same CV are the same!

Knowledge: We can solve the (lin. RP) exactly by the method of characteristics.

Strategy: Compute the left $(x<0)$ and right $(x>0)$ CV of the (lin. RP) and the (orig. RP).

Then equate the equations to find $F^{Roe}_{i+1/2}$.

orig. RP: we define $x_{i-1/2} \equiv x_L$

$x_{i+1/2} \equiv x_R$

(1). $$\int_{x_L}^{0} \int_{t^n}^{t^{n+1}} \partial_t(Q) + \frac{\partial f(Q)}{\partial x} \, dt \, dx$$

$$= \int_{x_L}^{0} Q(t^{n+1}, x) - Q(t^n, x) \, dx$$

$$+ \int_{t^n}^{t^{n+1}} \underbrace{f(Q(0,t))}_{f^{Roe}_{i+1/2}} - \underbrace{f(Q(x_L,t))}_{= f_L} \, dt$$

$$= \int_{x_L}^{0} Q(t^{n+1}, x) - Q(t^n, x) \, dx$$

$$+ (t^{n+1} - t^n) \cdot (f^{Roe}_{i+1/2} - f_L)$$

(2) $$\int_0^{x_R} \int_{t^n}^{t^{n+1}} \partial_t Q + \partial_x f(Q) \, dt\, dx$$

$$= \int_0^{x_R} Q(t^{n+1}, x) - Q(t^n, x) \, dx$$

$$+ \underbrace{\int_{t^n}^{t^{n+1}} \underbrace{f(Q(x_R, t))}_{= f_R} - \underbrace{f(Q(0, t))}_{f_{HLLC}^{Roe}} \, dt}_{(t^{n+1} - t^n) \cdot (f_R - f_{HLLC}^{Roe})}$$

(lin. RP.)

(3) $$\int_{x_L}^{0} \int_{t^n}^{t^{n+1}} \frac{\partial \bar{Q}}{\partial t} + \underbrace{\bar{A}(\tilde{Q}) \cdot \frac{\partial \bar{Q}}{\partial x}}_{(*)} \, dt\, dx$$

Trick (*).

$\bar{A}(\tilde{Q})$ is constant.

$$\bar{A}(\tilde{Q})\frac{\partial \bar{Q}}{\partial x} = \frac{\partial}{\partial x}\Big(\underbrace{\bar{A}(\tilde{Q})\,\bar{Q}}_{\equiv \bar{f}}\Big)$$

$$= \int_{x_c}^{0}\int_{t^n}^{t^{n+1}} \frac{\partial \bar{Q}}{\partial t} + \frac{\partial \bar{f}(\bar{Q})}{\partial x}\, dt\, dx .$$

$$= \int_{x_c}^{0} Q(t^{n+1},x) - Q(t^n,x)\, dx$$

$$+ \int_{t^n}^{t^{n+1}} \underbrace{\bar{f}(0,t)}_{\cancel{= f(Q}} - \underbrace{\bar{f}(x_c,t)}_{\cancel{f(Q_c)}}\, dt$$

$f^{\text{lin. RP}}_{i+1/2}$

$\left(= f^{\text{God}}_{i+1/2}\right)$

$$= \int_{x_C}^{0} Q(t^{n+1}, x) - Q(t^n, x)\, dx$$

$$+ (t^{n+1} - t^n) \cdot \left(\bar{f}^{l.RP}_{i+1/2} - f(Q_C) \right)$$

(4) in a ~~stai~~ similar way: we get

$$\int_0^{x_R} \int_{t^n}^{t^{n+1}} \frac{\partial \bar{Q}}{\partial t} + \underline{\underline{\bar{A}}}(\tilde{Q}) \cdot \frac{\partial \bar{Q}}{\partial x}\, dt\, dx$$

$$= \int_0^{x_R} \bar{Q}(t^{n+1}, x) - \bar{Q}(t^n, x)\, dx$$

$$+ (t^{n+1} - t^n) \cdot \left(\underbrace{\bar{f}_R}_{\cancel{\equiv f_R}} - \bar{f}^{l.RP}_{i+1/2} \right)$$

Using Roe's assumption, we get that

$$\underbrace{\int_{x_L}^{0} \bar{Q}(x, t^{n+1})}_{\text{(lin. RP)}} = \underbrace{\int_{x_L}^{0} Q(x, t^{n+1})}_{\text{(orig. RP)}}$$

we do the same for $\int_0^{x_R} \ldots$

Substract:

(1) - (3):

$$\underbrace{(t^{n+1} - t^n)\left(f^{Roe}_{i+1/2} - f_L\right)}_{(1)}$$

$$- (t^{n+1} - t^n) \cdot \left(\bar{f}^{\,l.RP}_{i+1/2} - \bar{f}_L\right) = 0$$

$$\Leftrightarrow \left(f^{Roe}_{i+1/2} - f_L\right) - \left[\cancel{\bar{A}(Q^{lin.RP}_{i+1/2})}\right]$$

$$- \left[\bar{A} Q^{l.RP.}_{i+1/2} - \bar{A} Q_L\right]$$

$$\boxed{\bar{f}(Q) := \bar{A}(\tilde{Q}) \cdot \cancel{\frac{\partial Q}{\partial x}}\, Q}$$

$$\Leftrightarrow \boxed{f^{Roe}_{i+1/2} = f_L + \bar{A}_i \cdot \left(Q^{l,RP.}_{i+1/2} - Q_L \right)} \quad (\mathrm{V})$$

Side note:

$$\frac{\partial \bar{Q}}{\partial t} + \frac{\partial}{\partial x} f(\bar{Q}) = 0$$

$$\Leftrightarrow \frac{\partial \bar{Q}}{\partial t} + \frac{\partial}{\partial x} (\bar{A} \bar{Q}) = 0$$

$$\Leftrightarrow \frac{\partial \bar{Q}}{\partial t} + \bar{A} \frac{\partial \bar{Q}}{\partial x} = 0$$

$(2)-(4)=0$

$$\underbrace{(t^{n+1}-t^n)(f_R - f^{Roe}_{i+1/2})}_{(2)}$$

$$-\Big[\underbrace{(t^{n+1}-t^n)\cdot(\bar{f}_R - \bar{f}^{l.R.P.}_{i+1/2})}_{=(4)}\Big] = 0$$

$$\Leftrightarrow \boxed{\begin{aligned} f^{Roe}_{i+1/2} &= f_R + (\bar{f}_R - \bar{f}^{l.R.P.}_{i+1/2}) = 0 \\ &= f_R + (\bar{A}(\tilde{Q})\cdot Q_R - \bar{A}(\tilde{Q})Q^{l.R.P.}_{i+1/2}) \\ &= f_R + \bar{A}(\tilde{Q})\cdot(Q_R - Q^{l.R.P.}_{i+1/2}) \end{aligned}}$$

(VI)

Average $\boxed{V}$ and $\underline{VI}$.

$$\boxed{f^{Roe}_{i+1/2} = \frac{1}{2}(f_L + f_R) + \bar{A}(\tilde{Q})\,\bar{Q}^{LRP}_{i+1/2} - \frac{1}{2}\bar{A}(\tilde{Q})\cdot(Q_R - Q_L)}$$

$\tilde{Q}$ (lin. state) is still unknown,

$\bar{Q}^{LRP}_{i+1/2}$ is known by the method of characteristics.

(HW) $\Leftrightarrow$

$$\boxed{f^{Roe}_{i+1/2} = \frac{1}{2}(f_L + f_R) \cancel{\bar{A}(\tilde{Q})} - \frac{1}{2}|\bar{A}(\tilde{Q})|(Q_R - Q_L)}$$

Problem: How to define $\tilde{Q}$?

$(\Leftrightarrow \bar{A}(\tilde{Q}))$

Weak formulation of Roe's solver

(Toumi, 1992)

Consider the generalized R.H. condition

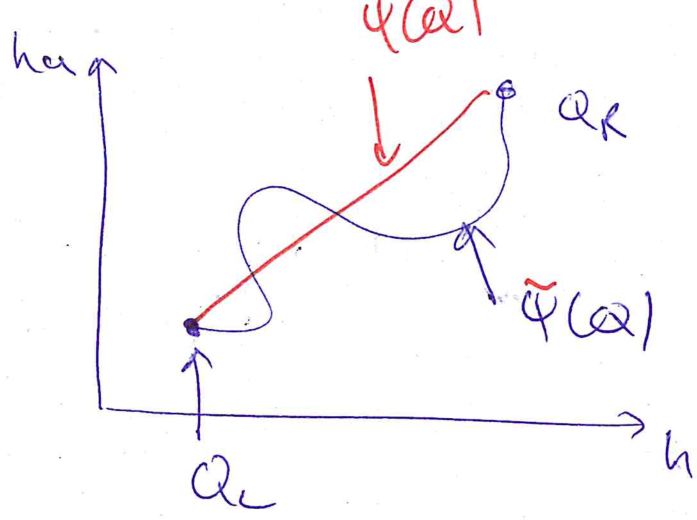
$$\underline{\underline{s(Q_R - Q_L)}} = \int_{Q_L}^{Q_R} \underline{\underline{A}}(Q)\, dQ$$

which is identical to the classical R.H. condition iff $\underline{\underline{A}} = \frac{\partial f}{\partial Q}$

$$\int_{Q_L}^{Q_R} A(Q)\, dQ = \int_{Q_L}^{Q_R} \frac{\partial f}{\partial Q}\, dQ = \underline{\underline{f(Q_R) - f(Q_L)}}$$

Exemple :

$Q = \begin{pmatrix} h \\ hu \end{pmatrix}$



$\cancel{\Psi(s)}$ $\Psi(p) = Q_L + p\,(Q_R - Q_L)$

$p \in [0,1]$

$$s(Q_R - Q_L) = \int_{Q_L}^{Q_R} A(Q)\, dQ$$

$$= \int_0^1 A(\psi) \frac{\partial \psi}{\partial p}\, dp$$

insert ψ.

$$\left[\frac{\partial \psi}{\partial p} = \frac{\partial}{\partial p} \left(Q_L + p(Q_R - Q_L) \right) = Q_R - Q_L \right]$$

$$= \int_0^1 A(\psi)\, [Q_R - Q_L]\, dp$$

$$= (Q_R - Q_L) \cdot \int_0^1 A(\psi)\, dp$$

We still seek to define

$$\bar{A}(\tilde{Q}).$$

Now we identify

$$\int_0^1 A(\Psi)\, dp =: \bar{A}(\tilde{Q})$$

In order to ~~be~~ for $\bar{A}(\tilde{Q})$ to be a proper Roe matrix, we needed to satisfy

- ~~hypo~~ hyperbolicity
- consistency
- conservation.

<u>Proof.</u>

$\to$ left out.

Path conservative FV - schemes:

Consider

$$\frac{\partial Q}{\partial t} + \partial_x f(Q) + \underline{\underline{NC}} \cdot \frac{\partial Q}{\partial x} = 0$$

with $\underline{\underline{NC}} \neq 0$.

We make sense out of that by writing the quasi linear form

$$\frac{\partial Q}{\partial t} + \underbrace{\left(\underline{\underline{\frac{\partial f}{\partial Q}}} + \underline{\underline{NC}} \right)}_{\equiv A} \cdot \frac{\partial Q}{\partial x} = 0$$

We follow the approach of the weak formulation of Roe's solver proposed by Toumi, while potentially violating the ~~conservation~~ conservation requirement

Remarks:

- This method requires a path $\Psi(\rho)$. Choosing $\Psi(\rho)$ as segment path is unphysical.
- "We conserve what we can"